

Update of the Electron and Muon g-Factors

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Abstract

I give an overview of the different contributions to the electron and muon anomalous magnetic moments in the Standard Model. Special emphasis is given to recent QED results as well as to the hadronic light-by-light scattering contribution to the muon anomaly.

Keywords: Magnetic moments of leptons, QED, QCD, Constituent Chiral Quark Model

1. Introduction

The g -factors of the leptons e , μ and τ are the dimensionless parameters which relate their spin \vec{s} to their magnetic moments $\vec{\mu}_l$:

$$\vec{\mu}_l = g_l \frac{e\hbar}{2m_l c} \vec{s}. \quad (1)$$

The anomalous magnetic moment a_l (anomaly for short), is the correction to the Dirac value $g_l = 2$:

$$\underbrace{g_l}_{\text{Dirac}} = 2 \left(1 + \underbrace{a_l = \frac{\alpha}{2\pi} + \dots}_{\text{anomaly}} \right), \quad l = e, \mu, \tau. \quad (2)$$

The lowest order contribution to the anomaly in powers of the fine structure constant α , the Schwinger term $\frac{\alpha}{2\pi}$, is the same for the three leptons. Higher order contributions become sensitive to the masses of the virtual particles exchanged and obey the following pattern:

- Heavy virtual particles with $M > m_l$ decouple as powers of $\frac{m_l^2}{M^2}$ modulated by powers of $\log\left(\frac{M^2}{m_l^2}\right)$ factors.
- Light virtual particles: e for $l = \mu, \tau$ and e, μ for $l = \tau$, bring in powers of log enhancements of the mass ratios.
- Virtual Hadronic Interactions and Electroweak Interactions become relevant at the present degree of experimental accuracy.

- Because of the relatively large value of the muon mass, the muon anomaly with its high precision measurement provides a sensitive probe for new fundamental physics.
- The tau anomaly is too poorly known experimentally to provide a test of the Standard Model.

2. Electron Anomaly and Fine Structure Constant

The best determination of the electron anomaly comes from experiments by the Harvard group [1] with the result:

$$a_e(\text{exp.}) = 1\,159\,652\,180.73(0.28) \times 10^{-12}, \quad (3)$$

which is a 0.24 ppb precision measurement! The Quantum Electrodynamics (QED) contributions of the massless class of Feynman diagrams (diagrams with virtual photons only as well as with virtual photons and fermion loops of the same flavour as the external particle, the electron in our case) have been evaluated up to tenth order

$$a_e(\text{QED} - \text{massless}) = \sum_{n=1}^{n=5} A_1^{(2n)} \left(\frac{\alpha}{\pi} \right)^n \quad (4)$$

with the following results:

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$$A_1^{(2)} = +0.5 \quad (5)$$

$$A_1^{(4)} = -0.328\,478\,965\,579\,193 \dots \quad (6)$$

$$A_1^{(6)} = +1.181\,241\,456 \dots \quad (7)$$

$$A_1^{(8)} = -1.9106\,(20) \quad (8)$$

$$A_1^{(10)} = +9.16\,(58). \quad (9)$$

The coefficient $A_1^{(2)}$ was first calculated by Schwinger in 1948 [2]. Ten years later Petermann and Sommerfield, independently, calculated $A_1^{(4)}$ [3, 4]. The sixth order term $A_1^{(6)}$ is also known *analytically* thanks to the dedicated work of Laporta and Remiddi [5]. The dots in $A_1^{(4)}$ and $A_1^{(6)}$ indicate that these are known numbers to any desirable accuracy. The eighth and tenth order coefficients $A_1^{(8)}$ and $A_1^{(10)}$ have been calculated *numerically* by Kinoshita and collaborators [6, 7]. The calculation of $A_1^{(10)}$ in particular involves 12672 Feynman diagrams! This is an extraordinary achievement after many years of dedicated effort. Notice the alternative signs for the coefficients $A_1^{(2n)}$: positive for n -odd and negative for n -even; an interesting pattern for which we have no simple explanation.

With $\frac{m_e^2}{m_\mu^2}$ and $\frac{m_\tau^2}{m_\mu^2}$ power corrections incorporated, as well as Hadronic Vacuum Polarization ($\sim 2 \times 10^{-12}$), Hadronic Light-by-Light Scattering ¹($\sim 3 \times 10^{-14}$) and Electroweak ($\sim 3 \times 10^{-14}$) corrections incorporated we have at present the following theoretical prediction:

$$a_e(\text{SM}) = 1\,159\,652\,181.82 \underbrace{(6)}_{\text{eighth}} \underbrace{(4)}_{\text{tenth}} \underbrace{(2)}_{\text{H-EW}} \underbrace{(78)}_{\alpha^{-1}(\text{Rb})} \times 10^{-12}, \quad (10)$$

which has a 0.67 ppb precision. Notice that the largest error here comes from inserting the Atomic Physics determination of α [8]. One can proceed otherwise: extract the value of α from the comparison between the predicted value of $a_e(\text{SM})$ and the Harvard measurement in Eq. 3, with the result:

$$\alpha^{-1}(a_e) = 137.035\,999\,1736 \underbrace{(68)}_{\text{eighth}} \underbrace{(46)}_{\text{tenth}} \underbrace{(26)}_{\text{H-EW}} \underbrace{(331)}_{\text{Harvard}}, \quad (11)$$

which represents a 0.25 ppb precision measurement and becomes at present the reference value of α for all the other QED observables (a_μ in particular).

¹See the estimate discussed in ref. [9].

3. Muon Anomaly

The present experimental world average determination of the muon anomaly is dominated by the latest BNL experiment of the E821 collaboration [10] with the result:

$$a_\mu(\text{E821} - \text{BNL}) = 116\,592\,089(54)_{\text{stat}}(33)_{\text{syst}} \times 10^{-11}, \quad (12)$$

which is a 0.54 ppm precision measurement.

At the level of the experimental accuracy, the QED contributions to a_μ from photons and leptons alone are very well known. The results, due to the work of many people ², are summarized in Table 1 below:

Table 1: QED Contributions (Leptons)
{ $\alpha^{-1} = 137.035\,999\,1736\,(342)\,[0.25\,\text{ppb}]$ }

CONTRIBUTION	RESULT IN POWERS OF $\frac{\alpha}{\pi}$
$a_\mu^{(2)}$	$0.5 \left(\frac{\alpha}{\pi}\right)$
$a_\mu^{(4)}(\text{total})$	$0.765\,857\,425\,(17) \left(\frac{\alpha}{\pi}\right)^2$
$a_\mu^{(6)}(\text{total})$	$24.050\,509\,96\,(32) \left(\frac{\alpha}{\pi}\right)^3$
$a_\mu^{(8)}(\text{total})$	$130.879\,6\,(63) \left(\frac{\alpha}{\pi}\right)^4$
$a_\mu^{(10)}(\text{total})$	$753.29\,(1.04) \left(\frac{\alpha}{\pi}\right)^5$
$a_\mu(\text{QED})$	$116\,584\,718.845\,(37) \times 10^{-11}$

The tenth order result $a_\mu^{(10)}(\text{total})$ in Table 1 is the one recently published by Kinoshita *et al* [11]. Again this is the result of very long impressive calculations improved during a long period of time.

The Standard Model contributions to a_μ with the Hadronic Vacuum Polarization of lowest order (HVP(lo)) and higher order (HVP(ho)), as well as from Hadronic Light-by-Light Scattering (HLbyL) and Electroweak (EW) effects incorporated are summarized in Table 2 below:

Table 2: Standard Model Contributions²

CONTRIBUTION	RESULT IN 10^{-11} UNITS
QED (leptons)	$116\,584\,718.85 \pm 0.04$
HVP(lo)[e^+e^-]	$6\,923 \pm 42$
HVP(ho)	-98.4 ± 0.7
HLbyL	105 ± 26
EW	153 ± 1
Total SM	$116\,591\,801 \pm 49$

The persistent $3.6\,\sigma$ discrepancy between the total SM result and the experimental value in Eq. 12 deserves

²For a recent review article where earlier references can be found see [12].

attention. In that respect, one observes that the largest errors in Table 2 come from the HVP and the HLbyL contributions. The size of these errors is of special concern in view of the future experiments at FNAL in the USA and at JPARC in Japan which plan to reduce the present experimental error from 0.54 ppm to 0.14 ppm. We examine these contributions in the next section.

4. Hadronic Vacuum Polarization and Hadronic Light-by-Light Scattering Contributions

The lowest order HVP contribution to the muon anomaly, shown in Fig. 1, has a well known integral representation in terms of the e^+e^- one photon annihilation cross-section into hadrons $\sigma_H(t)$, where t denotes the total CM energy squared:

$$a_\mu^{(\text{HVP})} = \frac{1}{4\pi^2} \int_{4m_\pi^2}^{\infty} dt \int_0^1 dx \frac{x^2(1-x)}{x^2 + \frac{t}{m_\mu^2}(1-x)} \sigma_H(t). \quad (13)$$

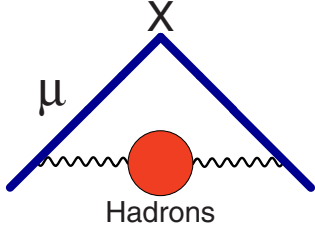


Figure 1:
Hadronic Vacuum Polarization Contribution.

This integral is in fact dominated by the contributions from the low- t region. Its determination has been improving through the years thanks to the advent of more and more refined measurements; the latest coming from the BaBar and Belle facilities. The error here is likely to be reduced in the near future.

Contrary to the HVP contribution, the Hadronic Light-by-Light Scattering contribution (HLbyL) shown in Fig. 2 cannot be written as an integral over experimentally accessible observables. The calculation from theory requires the knowledge of Quantum Chromodynamics (QCD) contributions at all energy scales, something which we don't know how to do at present except via numerical lattice QCD evaluations which, unfortunately, are difficult to implement in this case³.

³There are, however, promising lattice projects under study, see e.g. ref. [13].

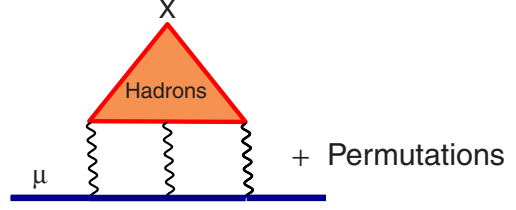


Figure 2:
Hadronic Light-by-Light Scattering Contribution.

Things, however, are not as hopeless as all that. We know some important constraints from QCD which the hadronic models one has to resort to, to evaluate this contribution, have to obey.

- *Chiral Limit and Large- N_c limit of QCD*

It is well known that the hadronic realization of QCD in the sector of the light u, d, s quarks is governed by the phenomena of *spontaneous chiral symmetry breaking and confinement*. This implies the existence of a *Mass Gap* between the pseudoscalar (*Goldstone-like*) particles and the other hadronic particles (the ρ being the lowest one in Nature). In the limit where this *Mass Gap* is considered to be large, and to leading order in the $1/N_c$ -expansion (N_c is the number of colours and the expansion in question is a topological expansion of phenomenological relevance), the contribution of the HLbyL scattering to the muon anomaly is dominated by the diagrams shown in Fig. 3 where only one pion propagator appears. The lead-

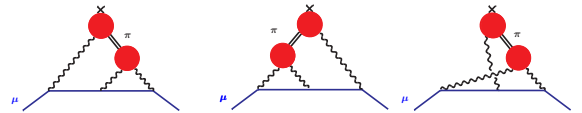


Figure 3:
Dominant HLbyL Scattering Contribution.

ing result in this limit is known analytically [14]:

$$a_\mu^{(\text{HLbyL})}(\pi^0) = \left(\frac{\alpha}{\pi}\right)^3 N_c^2 \frac{m_\mu^2}{48\pi^2 f_\pi^2} \left[\log^2 \frac{M_\rho}{m_\pi} + O\left(\log \frac{M_\rho}{m_\pi}\right) + O(1) \right]. \quad (14)$$

Notice that the leading order approximation in the $1/N_c$ -expansion is here necessary because it selects the one-pion exchange contribution; pion loops can only appear at the next-to-leading level.

This result is phenomenologically relevant because it can be used as a check of Hadronic Model Calculations: letting the hadronic masses of a model become very large one must find an answer compatible with the analytic result in Eq. 14 when M_ρ is also taken to be large. The fact that the HLbyL contribution in this limit is positive was crucial in fixing the overall sign of $a_\mu^{(\text{HLbyL})}$ [15]. Numerically, for the physical values of the constants in Eq. 14 one finds

$$\left(\frac{\alpha}{\pi}\right)^3 N_c^2 \frac{m_\mu^2}{48\pi^2 f_\pi^2} \log^2 \frac{M_\rho}{m_\pi} = 95 \times 10^{-11}, \quad (15)$$

which is within the ballpark of the hadronic model determinations. The ρ -mass, however, is too close to the π -mass to take seriously this number which may have large corrections from the $O(\log \frac{m_\rho}{m_\pi})$ and $O(1)$ terms. In fact, it is in principle possible to fix the coefficient of the $O(\log \frac{m_\rho}{m_\pi})$ term. It can be shown [16, 17] that the unknown contribution to this coefficient is related to the $\pi^0 \rightarrow e^+e^-$ decay rate, albeit with the radiative corrections included. The $O(1)$ term is, however, not fixed by symmetry restrictions which limits the use of Eq. 14 at the level of the required accuracy.

- *Short-Distance Constraint from the OPE in QCD*

The constraint in question comes from a clever observation by Melnikov and Vainshtein [18]. The three momenta k_1, k_2, k_3 in the light-by-light subdiagram of the HLbyL scattering contribution to the muon anomaly (see Fig 4 below) form a tri-

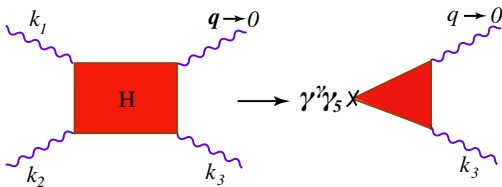


Figure 4:

The OPE Constraint of Melnikov and Vainshtein.

angle. When $k_1^2 \approx k_2^2 \gg k_3^2$, and $k_1^2 \approx k_2^2 \gg m_\rho^2$ in this triangle one can apply the OPE in the two

vector currents which carry hard momenta with the result:

$$\int d^4x_1 \int d^4x_2 e^{-ik_1 \cdot x_1 - ik_2 \cdot x_2} J_\gamma(x_1) J_\rho(x_2) = \frac{2\epsilon_{\gamma\rho\delta\gamma} \hat{k}^\delta}{\hat{k}^2} \int d^4z e^{-ik_3 \cdot z} J_5^\gamma(z) + O\left(\frac{1}{\hat{k}^3}\right), \quad (16)$$

where $j_5^\gamma = \sum_q Q_q^2 \bar{q} \gamma^\gamma \gamma_5 q$ is the axial current with the different flavors weighted by squares of their electric charges and $\hat{k} = (k_1 - k_2)/2 \approx k_1 \approx -k_2$. As illustrated in Fig. 4 this OPE reduces the HLbyL amplitude, in the special kinematics under consideration, to the AVV triangle amplitude which is an object for which we have a much better theoretical insight. This observation has interesting phenomenological implications:

- At large $k_{1,2}$ the Pseudoscalar (and Axial-Vector) exchanges dominate.
- The AVV limit also implies that the $\mathcal{F}_{\pi^0 \gamma^* \gamma^*}(k^2, k^2)$ form factor at the vertices of Fig 3 must fall as $1/k^2$.

Unfortunately, the two asymptotic QCD constraints discussed above are not sufficient for a full model independent evaluation of $a_\mu^{(\text{HLbyL})}$. This explains the relatively large error of $\pm 26 \times 10^{-11}$ for this contribution in Table 2 above.

Most of the last decade calculations of $a_\mu^{(\text{HLbyL})}$ found in the literature ² are compatible with the QCD chiral constraints and the large- N_c limit discussed above. They all incorporate the π^0 -exchange contribution modulated by $\pi^0 \gamma^* \gamma^*$ form factors correctly normalized to the Adler, Bell–Jackiw point-like coupling. They differ, however, on whether or not they satisfy the particular OPE constraint discussed above, and in the shape of the vertex form factors which follow from the different models. In spite of the different choices of these form factors there is, within errors, a reasonable agreement among the final results. An exception is the calculation reported in ref. [19] using a model inspired on a Dyson–Schwinger approach which, however, as we shall see contradicts generic properties which emerge from the Constituent Chiral Quark Model which we next discuss.

5. The Constituent Chiral Quark Model

I have recently emphasized the need of a simple *reference model* to evaluate the various hadronic contributions to a_μ within the same framework, and use it as

a yardstick to compare with the more detailed evaluations in the literature. The *reference model* which we have proposed [20] is based on the Constituent Chiral Quark Model (C χ QM) of Manohar and Georgi [21] in the presence of $SU(3)_L \times SU(3)_R$ external sources. It is an effective field theory which incorporates the interactions of the Nambu–Goldstone modes (the low-lying pseudoscalars) of the spontaneously broken chiral symmetry, to lowest order in the chiral expansion and in the presence of chirally rotated quark fields which have become massive. As emphasized by Weinberg [22] the effective Lagrangian in question is renormalizable in the Large- N_c limit and, as shown in [23], the number of the required counterterms is minimized for the choice $g_A = 1$ of the axial coupling of the constituent quarks to the pseudoscalars. The model has its limitations but, as discussed in ref. [23], there is an exceptional class of low-energy observables for which the predictions of the C χ QM can be rather reliable. This is the case when the leading short-distance behaviour of the underlying Green’s function of a given observable is governed by perturbative QCD. The contributions to the muon anomaly from Hadronic Vacuum Polarization, from the Hadronic Light-by-Light Scattering and from the Hadronic $Z\gamma\gamma$ vertex (provided that $g_A = 1$) fall in this class.

Figure 5 below shows the prediction of the C χ QM for the HVP contribution to the muon anomaly as a function of the constituent quark mass M_Q which, with $g_A = 1$, is the only free parameter. The curve (a) in

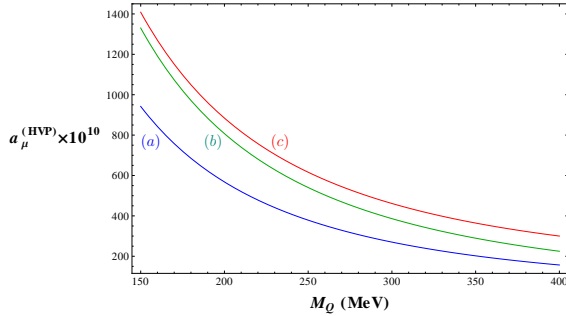


Figure 5:
The HVP Contribution in the C χ QM.

Fig. 5 is the contribution using the spectral function of the C χ QM; curve (b) the same contribution as in curve (a) but with gluonic corrections included and curve (c) the same contribution as in curve (b) but with $1/N_c$ sub-leading $\pi^+\pi^-$ and K^+K^- contributions incorporated as predicted in the C χ QM. The comparison between this

prediction and the phenomenological determination of $a_\mu^{(\text{HVP})}$ shows that fixing M_Q in the range

$$M_Q = (240 \pm 10) \text{ MeV}, \quad (17)$$

reproduces the phenomenological determination within an error of less than 10%. This error, however, only reflects the phenomenological choice that we have made to fix M_Q , it does not include the systematic error of the C χ QM itself. As shown in ref. [20], with M_Q fixed in this range, the higher order HVP contributions, as well as the electroweak Hadronic $Z\gamma\gamma$ contribution are reproduced rather well with the C χ QM.

When examining the HLbyL scattering contribution to the muon anomaly in the C χ QM, one finds that there are two competing contributions: one is the π^0 -exchange diagrams in Fig. 6 where the circles there are constituent quark loops, the other one the irreducible constituent quark loop in Fig. 7.

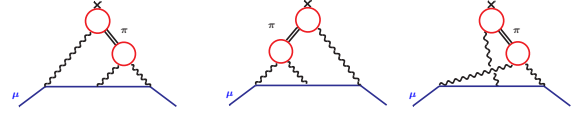


Figure 6:
The π^0 -Exchange Contribution in the C χ QM.

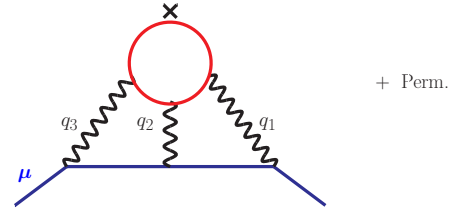


Figure 7:
The Quark Loop Contribution in the C χ QM.

An interesting feature which emerges from the calculation is the balance between the *Goldstone Contribution* and the *Quark Loop Contribution*. Indeed, as the constituent quark mass M_Q gets larger and larger, the *Goldstone Contribution* dominates: asymptotically, for large M_Q values it behaves (as expected) like

$$a_\mu^{(\text{HLbyL})}(\text{GC}) = \left(\frac{\alpha}{\pi}\right)^3 N_c^2 \frac{m_\mu^2}{48\pi^2 f_\pi^2} \times \left[\log^2 \frac{M_Q}{m_\pi} + \mathcal{O}\left(\log \frac{M_Q}{m_\pi}\right) \right], \quad (18)$$

while for M_Q smaller and smaller it is the *Constituent Quark Loop Contribution* which dominates: asymptotically, for small M_Q values (though still with $M_Q > m_\mu$) it behaves like

$$a_\mu^{\text{HLbyL}}(\text{CQL}) = \left(\frac{\alpha}{\pi}\right)^3 N_c \left(\sum_{q=u,d,s} \mathcal{Q}_q^4 \right) \times \left\{ \left[\frac{3}{2} \zeta(3) - \frac{19}{16} \right] \frac{m_\mu^2}{M_Q^2} + \mathcal{O} \left(\frac{m_\mu^4}{M_Q^4} \log^2 \frac{m_\mu^2}{M_Q^2} \right) \right\}. \quad (19)$$

These features are illustrated by the plot of the total $a_\mu^{\text{HLbyL}}(\text{C}\chi\text{QM})$ versus M_Q shown in Fig. 8. In fact the plot also shows that the value of $a_\mu^{\text{HLbyL}}(\text{C}\chi\text{QM})$ is quite stable for a rather large choice of reasonable M_Q values.

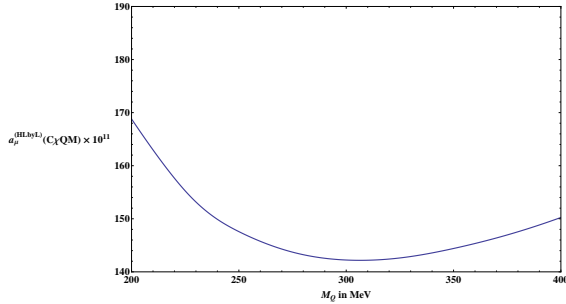


Figure 8:
The HLbyL Contribution in the $\text{C}\chi\text{QM}$.

The $\text{C}\chi\text{QM}$ result contradicts what is reported in ref. [19] where the equivalent contribution to the *constituent quark loop* is found to be: $(136 \pm 59) \times 10^{-11}$ i.e. much larger than the contribution found by the same authors for the π^0 -exchange: $(81 \pm 12) \times 10^{-11}$ which, within errors, is compatible with other phenomenological determinations. This casts serious doubts about the compatibility of the model used in ref. [19] (or perhaps of their calculation⁴) with basic QCD features encoded in the $\text{C}\chi\text{QM}$.

We conclude that, in the absence of more refined calculations, the number quoted in Table 2 for the light-by-light scattering contribution to the muon anomaly still represents a valid estimate. In fact, there is a recent

independent analysis in ref. [25] which confirms that estimate.

Acknowledgements

I wish to thank David Greynat for a pleasant collaboration on the $\text{C}\chi\text{QM}$ calculations reported here and, together with Marc Knecht and Laurent Lellouch, for useful discussions and suggestions.

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⁴After the completion of this mini review, there has appeared a new version of this model in the archives [24] with a smaller result for the quark loop contribution: $(96 \pm 2) \times 10^{-11}$. Notice that the error here does not include the systematic error of the model and the calculation is claimed to be incomplete as yet.